Proving the Quadratic Formula

Why Does It Work?

Lauren Martin

History of Mathematics

Jennifer McCarthy and Annie Cox

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Abstract

The quadratic formula was created by al-Khwārizmī, an Arab mathematician from the Ninth Century. The modern form of the formula is . The quadratic formula is a method used to solve for *x* in quadratic equations. To use the quadratic formula, you must take the coefficients and constants and plug them into their corresponding positions in the formula. Another method of solving for *x* is completing the square, which is used to prove the quadratic formula. To complete the square you must take the middle term's coefficient and half, square, and add it to each side of the equation. Taking the standard quadratic form and completing the square on it results in the quadratic formula.

History

Mathematics in the Ninth Century was not the same as it is now. Back then, zero was not even considered a number, just a place holder. The concept of “nothing” being “something” had not been thought of, yet. Also, negative numbers as answers were immediately discarded, because mathematics was mainly used to find measurements and negative lengths and widths were impossible. Al-Khwārizmī was one of the first to recognize that quadratic equations could have two roots, or answers. Though he only used the positive answers, the fact that he considered negative numbers as answers was a big step in history (Berlinghoff, 2002).

Muhammad Ibn Mūsa al-Khwārizmī was one of the greatest Arab mathematicians of the Ninth Century. Named after his birth city, Khwārizm, located in Uzbekistan, al-Khwārizmī moved to Baghdad later in life where he studied at The House of Wisdom, a library designed for research and translation. He studied all different areas of mathematics, such as the decimal place value system and the relevance of zero. However, al-Khwārizmī’s main contribution was his book on algebra, “al-jabr w’al- muqābala.” Written around the year 825, the book mainly discusses linear and quadratic equations, but also contains geometry. After being translated to Latin many years later, the word “al-jabr” became “algebra,” the English word we use today (Berlinghoff, 2002).

When writing his work down, al-Khwārizmī had to write everything out in words because symbols for algebra had not been invented. Historians recognized that al-Khwārizmī used the word “shai” for *x* in his studies. “Shai” was translated into “res” in Latin, which means “thing.” (Berlinghoff, 2002)

Here is an example of one of al-Khwārizmī's problems translated by John of Seville in the Twelfth Century:

"It is asked, therefore, what thing together with ten of its roots or what is the same, ten times the root obtained from it, yields thirty-nine." (Berlinghoff, 2002, p. 95)

We would write this as $x^2 + 10x = 39$ using our current symbolism for algebraic equations (Berlinghoff, 2002).

Proof of the Quadratic Formula

While studying quadratic equations, al-Khwārizmī discovered a relatively easy formula for solving them. The formula can be used to solve any quadratic equation, which is why it is so widely used. Even schools have a mnemonic device in the form of a song so students will always remember the quadratic formula. The formula consists of simply plugging in numbers from the equation and then simplifying. The main form of the formula is

Al-Khwārizmī's original form was much more complicated and revised for better utilization (Berlinghoff, 2002). The only way the formula works, however, is if the equation is in the basic quadratic form, which is:

$$ax^2 + bx + c = 0$$

Here is an example of how to solve a quadratic equation by the quadratic formula:

$$x^2 - 5x - 6 = 0$$

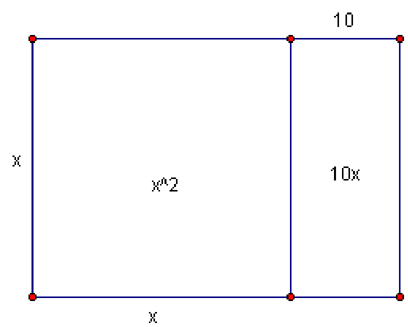
a = 1   *b = -5*   *c = -6*

Now, simply plug those numbers into the quadratic formula and simplify.

$x =$ $x = 6, -1$

The quadratic formula can be a practical tool when trying to solve equations, but why does it work? This can be answered by taking another method of solving quadratic equations and applying it to the basic quadratic form. This method is called completing the square.

In ancient history, mathematicians had no calculators or advanced computing devices, so they actually had to take a square and complete it geometrically, as shown in the diagram below.



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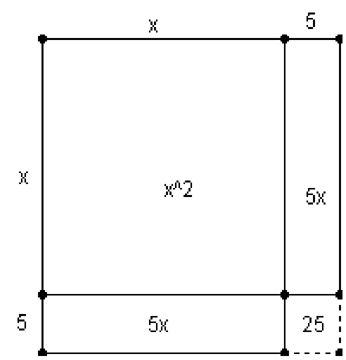

(Berlinghoff, 2002)

The purpose of the entire process is to literally complete the square by finding out how many small squares would fit inside the missing part of the larger squares. The problem originally started out with *10x* and by dividing by 2 and placing each rectangle on the sides, a 5x5 square was created. Therefore, 25 small squares would complete the square. By adding 25 to the square, 25 must be added to the total value of the square to sustain the balance of the equation.

Today, this process has a specific algorithm for finding the answer. The modern method of completing the square is how the quadratic formula can be proved. Let us see what happens when you apply this method to the basic quadratic form and solve for *x*.

Step 1 Start out with the standard form for a quadratic equation.

$$ax^2 + bx + c = 0$$

Step 2 Begin the process of completing the square by subtracting *c* from both sides.

$$ax^2 + bx = -c$$

Step 3 Divide by *a* to get the coefficient of x^2 to be 1.

Step 4 Divide *b* by 2, square it, and add to both sides.

$$x^2 + (–) + (—)^2 = (—)^2 - –$$

Step 5 Now that the left side of the equation is a perfect square trinomial, you can factor.

$$(x + —)^2 = — + (—)^2$$

Step 6 Take the square root of the whole equation.

$$— \quad — \quad — \rightarrow x + — = \quad — \quad —$$

Step 7 To solve for *x*, subtract — from both sides.

$$x = - \ — \quad — \quad –$$

Step 8 Distribute the square.

$$x = -$$

Step 9 Find a common denominator and add everything under the radical.

$$\rightarrow$$

Step 10 Insert step 9 under the radical.

$$x = -$$

Step 11 Find the square root of the denominator.

$$x = -$$

Step 12 Simplify.

$$x =$$

The quadratic formula is basically a shortcut for solving quadratic equations by completing the square. Completing the square on a standard quadratic equation and then solving for *x* results in the quadratic formula. Thus, substituting coefficients into the quadratic formula is the same as completing the square on the original quadratic equation.

References

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