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The Magical Menagerie of Mathematics:

Proving the Pythagorean Theorem

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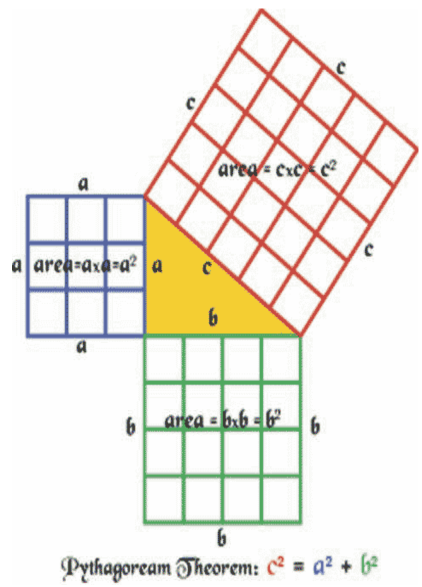
Abstract

Pythagoras made many contributions to the mathematical, scientific, and philosophical developments occurring during his time. He was born in the early sixth century BC on Samos. Although he is famous for mathematics and science now, in his time and even a century and a half later, he was known for his philosophical theories dealing with aspects such as life and death (Joost-Gaugier, 2006). He was said to be one of the greatest and most influential mathematicians of his time although little to no information on his life or teachings is recorded (Joost-Gaugier, 2006). Still his influence is eminent; one of his most acclaimed and famous contributions is the Pythagorean theorem. This theorem is used to express the relationship between the sides of a right triangle and its hypotenuse. The Pythagorean theorem states that a square drawn on the hypotenuse (the longest side) of a right triangle is equal in area to the sum of the squares drawn on the other two sides (the legs). This is shown geometrically by comparing areas of shapes, and algebraically by the equation: $c^2 = a^2 + b^2$. Thousands of proofs have been found since Pythagoras' discovery and a variety of proofs exist. Proofs for the theorem range from methods of dissection and using the properties of sine.



By researching different proofs and attempting an original proof by rearrangement, I was able to better understand the relationship the Pythagorean theorem illustrates. Although some think that since the theorem bears Pythagoras' name it is his original proof, it turns out that the theorem has been around for centuries before him.

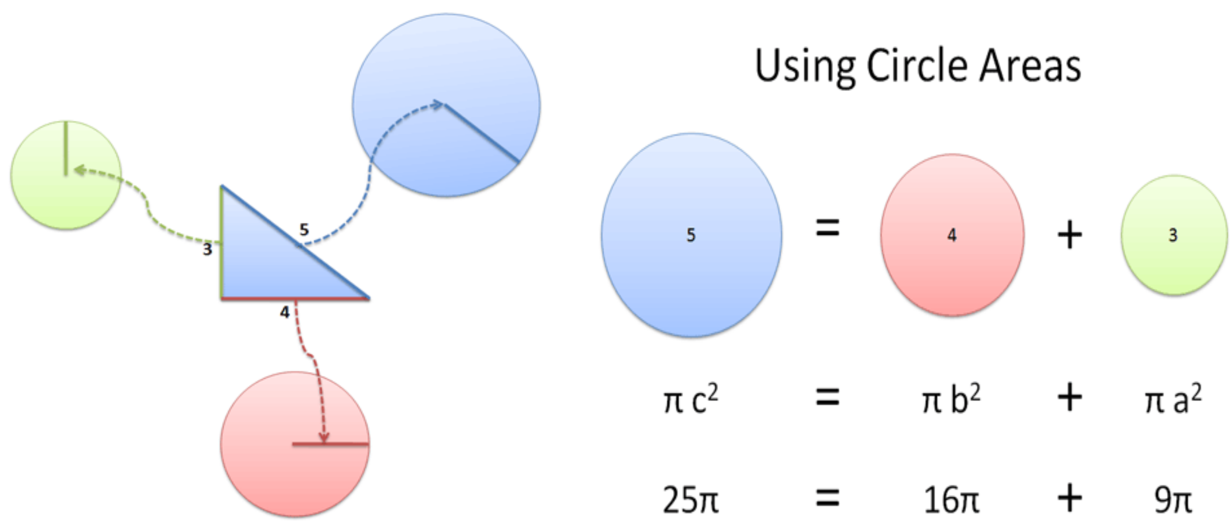
The Pythagorean Theorem
Background

While this theorem has been made famous by and been accredited to Pythagoras, various other claims to the origin of this equation have been made. One source claims that the proposition now bearing Pythagoras' name was discovered on a Babylonian tablet (c. 1900 - 1600 BC) 1000 years before Pythagoras' time. Pythagoras was captured by the Persians and taken captive to Babylon in 525 BC, where he spent five years before miraculously escaping to Samos (Bogomolny). He is the first credited with actually having proved it using the deductive proof sequence of geometry. It is evident that Pythagoras was not the first or only mathematician developing this proof. History records confirm that before, during, and after Pythagoras' time, mathematicians from other countries and time periods were also deducing the same theorem.

Another source lists the Pythagorean theorem by another name: The Gougu Theorem (Math Maze, 2006). The Chinese argue that although the theorem is named after Pythagoras, he was not the first to discover it. A pattern is present in descriptions of the theorem depending on the source, but there is one facet that seems to be a common factor; all sources acknowledge that Pythagoras was not the first to develop or discover this theorem whereas most think he was the mathematician who engendered it.

The Pythagorean theorem is now used to find missing side lengths, but is only applicable to right triangles. Aside from being generally related to geometry and triangles, the Pythagorean theorem can be found applicable in other circumstances such as in social networks, surface area, computer science, and physics. For example, in social networks, Metcalfe's Law says the value of a network is about n^2 (the number of relationships). In terms of value, a Network of 50M = Network of 40M + Network of 30M. The 2nd and 3rd networks have 70M people total, but they aren't a coherent whole. The network with 50 million people is as valuable as the others combined (Azad, 2009). This also illustrates how the additive property and the multiplicative property differ. This relation is also present in Computer Science. Some programs with n inputs take n^2 time to run. In terms of processing time: 50 inputs = 40 inputs + 30 inputs. Similarly to the previous example, 70 elements spread amongst two groups can be sorted as fast as 50 items in one group (Azad, 2009). The Pythagorean theorem helps show how sorting 50 combined elements can be as slow as sorting 30 and 40 separate ones.

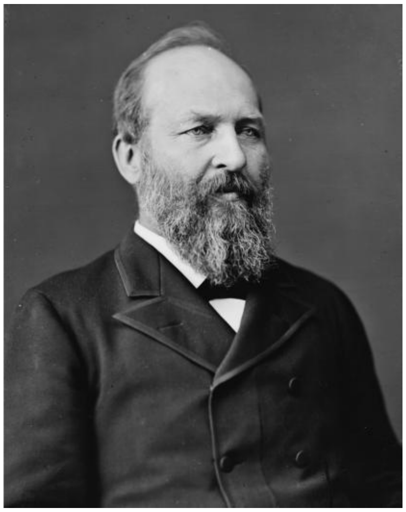
Another application of the Pythagorean theorem is in the area of circles. The area of a circle is πr^2. So, in terms of areas of circles: Area of radius 5 = area of radius 4 + area of radius 3. The following illustration depicts the similarities between the ratios of right triangles, radii of circles and the circles' areas.

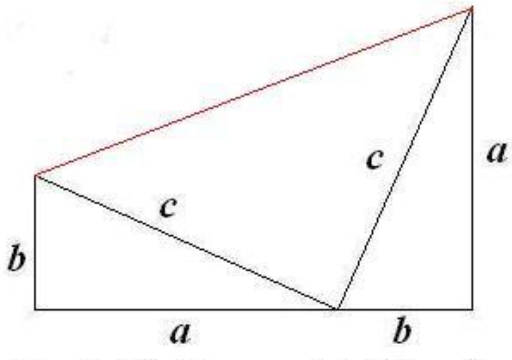


Circles with radii correspondent with the sides of a 3,4,5 triangle share the relationship of the Pythagorean theorem. The areas of circles clearly show the Pythagorean relationship.

This relationship is also applicable in physics. The kinetic energy of an object with mass m and velocity v is equal to 1/2 m v^2. In terms of energy, Energy at 500 mph = Energy at 400 mph + Energy at 300 mph. With the energy used to accelerate one bullet to 500 mph, we could accelerate two others to 400 and 300 mph. (Azad, 2009)

Evidently various mathematicians all over the world dealt with this theorem, including a United States President who created his own proof for the Pythagorean theorem. James Abram Garfield (1831-1881) was sworn in as the twentieth United States President on March 4, 1881. He was the first president to develop a proof for the Pythagorean theorem, which he wrote in 1876 while still a member of the House of Representatives (Smith, 1996). Garfield's proof incorporates the fact that the area of a trapezoid is half the product of its altitude multiplied by the sum of the lengths of its parallel bases. Garfield used three right triangles to form a trapezoid where the sum of the trapezoid is the sum of the areas of the triangles. His proof was published in the New England Journal of Education on April 1, 1876 (Smith, 1996).





Garfield's Trapezoidal Proof

Garfield's proof:

$$\tfrac{1}{2}(a+b)(a+b) = \tfrac{1}{2}ab + \tfrac{1}{2}ab + \tfrac{1}{2}c^2$$

$$(a+b)(a+b) = ab + ab + c^2$$

$$a^2 + 2ab + b^2 = 2ab + c^2$$

$$a^2 + b^2 = c^2$$

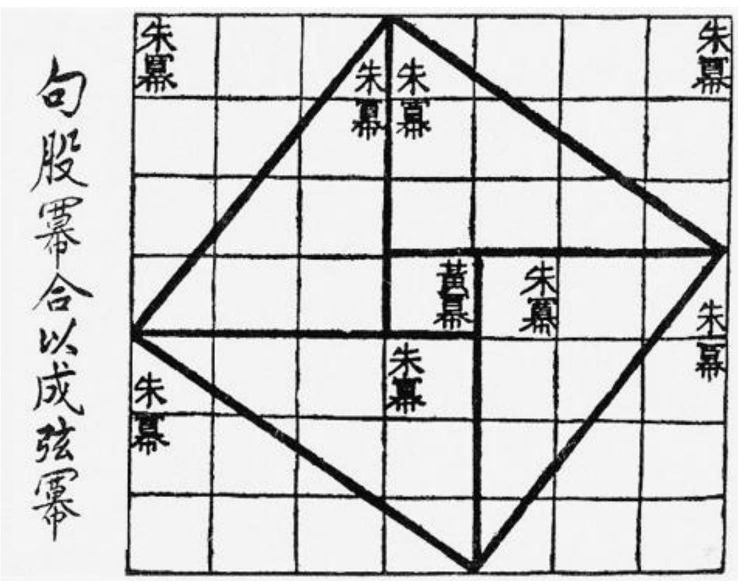
Several hundred different proofs of the Pythagorean theorem have been recorded, so Garfield's proof doesn't rank amongst the greatest of mathematical accomplishments. Still, it is historically interesting and quite eloquent (Smith,1996).

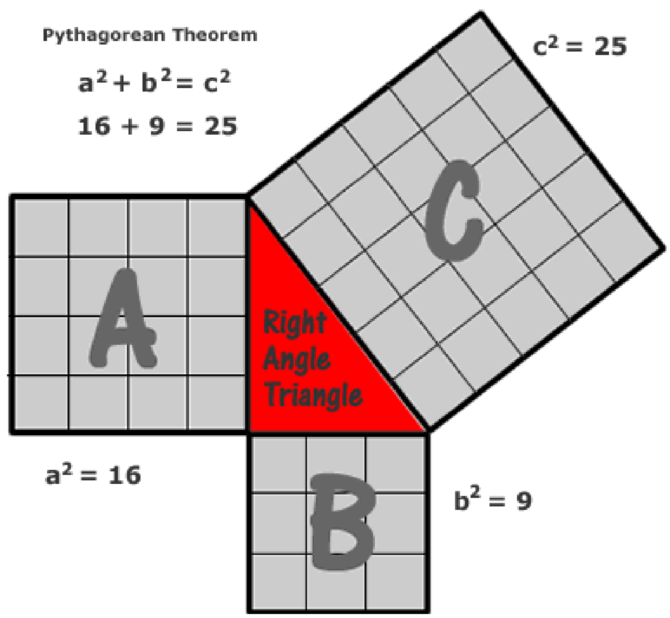
Research Dilemma

The Pythagorean proof has been around for centuries and is one of the most widely recognized theorems in mathematics. Since thousands of variations of the theorem exist, what exactly constitutes the Pythagorean theorem? When methods from different countries and time periods are compared, what similarities exist in the proofs? What differences make the proofs original? Proofs for the Pythagorean theorem range from methods of dissection and rearrangement, to algebraic expression, geometry based proofs, to using the radii and areas of circles to draw conclusions. In the case of circles, there is usually a common ratio which illustrates the Pythagorean relation is present. At times, the ratio 3:4:5 is present, what variations of this ratio or other ratios produce a concise proof? Do the variant sizes effect producing an effective Pythagorean proof? An analysis of variant sizes, new created shapes of the proof in comparison to that of the original shapes, can prove or disprove a theorem. Can any method of dissection and rearrangement prove the Pythagorean theorem as long as the areas of the two shapes are equal?

Methods

One proof of the Pythagorean theorem is called the Gougu Proof. It uses four, 3, 4, 5 right triangles. Although all the calculations are in Chinese, the mathematical result is the same making it an accurate proof for the Pythagorean or “Gougu” theorem.





The image on the left is the illustration for the Gougu Theorem. The Illustration on the right is the Pythagorean theorem, which in turn proves the Gougu theorem. The Gougu Theorem has four 3, 4, 5 right triangles which all correspond with the triangles in the proof on the right. An example of how the Chinese would apply this proof is as follows:

Given: kou = 3 ch'ih, ku = 4 ch'ih. What is the length of hsien?

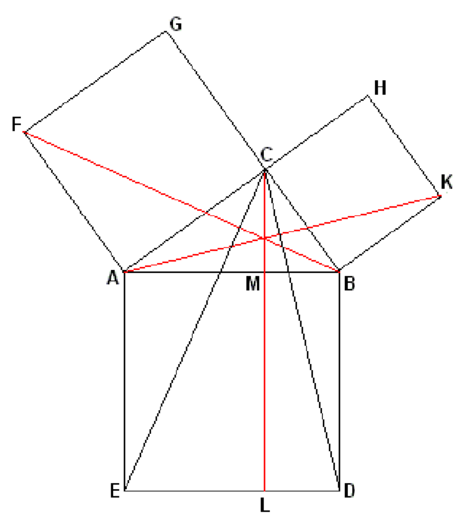
Answer: hsien = 5 ch'ih

Method: Add the square of kou and ku. The square root of the sum is equal to hsien. (Swetz; Kao, 1977, p17).

The existence of this proof proves that Pythagoras was not the only one developing this proof, and that its development occurred at all over the world and in different times through history.

The Chinese would have worded a math problem pertaining to the Pythagorean theorem like this: “There grows in the middle of a circular pond 10 feet in diameter a reed which projects 1 foot out of the water. When it is drawn down it just reaches the edge of the pond. How deep is the water?” (Smith, 1996, p.7) Interesting enough, the theorem can be seen written differently depending on the context it needed to fit for better understanding in that region. Word problems are like analogies created so the reader can better understand and apply themselves.

Another proof is called The Bride’s Chair. In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.



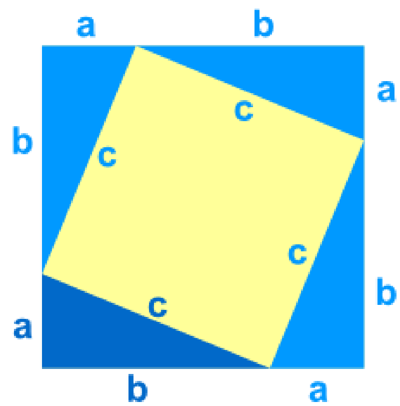
Given Proof

1. draw lines segments BF and AK	1. two points determine a unique line
2. through C construct CL parallel to BD intersecting AB at M	2. thru an external pt. there is 1 parallel
3. angle BAC = angle BAC	3. self-identity
4. angle FAC = angle BAE	4. all right angles are =
5. angle FAC + angle CAB = angle CAB + angle BAE	5. if =s are added to =s, the sums are =
6. ∆ABF = ∆AEC	6. SAS (congruence by side-angle-side)
7. area ∆ABF with altitude AC on base AF = ½ area ☐ACGF	7. area ∆ = ½ area Ƨ same base & alt.
8. area ∆AEC with altitude AM on base AE = ½ area AELM	8. area ∆ = ½ area Ƨ same base & alt.
9. ½ area ☐ACGF = ½ area ☐AELM	9. transitive property of =
10. area ☐ACGF = area ☐AELM	10. if =s multiplied by =s products are =
11. similarly area ☐BCHK = area ☐BDLM	11. counterparts of reasons 3. - 10.
12. but area ☐AELM + area ☐BDLM = area ☐ABDE	12. the whole = the sum of its parts
13. area ☐ABDE = area ☐ACGF + area ☐BCHK	13. =s may be substituted for =s

(Astrology Reading)

This proof takes a geometry-based approach. It analyzes sides and angles instead side lengths to prove the Pythagorean theorem.

This proof is all algebra. By using the side lengths, area of the whole, and the area of the four triangles and smaller square, the Pythagorean theorem can be proved.



Area of Whole Square

It is one big square as a whole, with each side having a length of a+b, so the total area is:

$$A = (a+b)(a+b)$$

Area of The Pieces

Now let's add up the areas of all the smaller pieces:
First, the smaller (tilted) square has an area of

$$A = c^2$$

And there are four triangles; each one has an area of

$$A = \tfrac{1}{2}ab$$

So all four of them combined is

$$A = 4(\tfrac{1}{2}ab) = 2ab$$

So, adding up the tilted square and the 4 triangles gives:

$$A = c^2+2ab$$

Both Areas Must Be Equal

The area of the large square is equal to the area of the tilted square and the 4 triangles. This can be written as:

$$(a+b)(a+b) = c^2+2ab$$

NOW, let us rearrange this to see if we can get the pythagoras' theorem:

Start with:

$$(a+b)(a+b) \quad = \quad c^2 + 2ab$$

Expand (a+b)(a+b):

$$a^2 + 2ab + b^2 \quad = \quad c^2 + 2ab$$

Subtract "2ab" from both sides:

$$a^2 + b^2 = \quad c^2$$

Results

The thousands of proofs for the Pythagorean theorem, each different and correct, show just how widely applicable the theorem is. While theorems range from methods of dissection, to algebraic proofs, to extensive geometric proofs, the end result of each comes out to the same. What we know about the Pythagorean theorem today is a culmination of efforts by mathematicians spanning throughout the world and as far back as centuries ago.

Although each proof is original, it can still be minor adaptation of another proof. That's why some proofs seem very similar to each other. Of course since they all aim to prove the Pythagorean theorem, they all undergo similar processes. One thing most proofs have in common is that they all incorporate right triangles. Where some might use angle bisectors and perpendicular bisectors, other might use angle and side theorems, but all in reference to right triangles. Even when dealing with circles, a right triangle can be created with the radius to show relationships between triangles and circles.

The theorem itself is not just applicable in mathematics. Whereas most people have the theorem imprinted in their memory bank from Geometry class, the Pythagorean proof can be used in a variety of circumstances. Physics, Social Networking, and relationships between shapes and objects and nature can be applied to the Pythagorean theorem.

From studying the origins, applications, and development of the Pythagorean theorem, it has become evident that in order to better understand and be able to use and apply a theorem, or any other useful tool, one must understand how it developed. Anyone can be given a formula in class and apply it, but it takes one with true insight to be able to produce the formula through example. Only someone with true understanding of how things came to function can better apply it and fully understand how to apply it, but why.

Conclusions/Extensions

Based on the number of proofs that exist for the Pythagorean theorem, one may wonder how many more proofs are still undiscovered. Even further, how many proofs are still undiscovered for other theorems in mathematics and even science. The broad world of the Pythagorean theorem, its many proofs and applications demonstrates possibilities in math and science. It also shows how there must be people with the drive to prove things they may not understand. And even more persistent people to fail and try again. How things work, and how humanity is able to create accommodations for the use of tools, exemplifies the power there is in knowledge. All the accomplishments that come with the developments of effective proofs and theorems is almost magical. But that’s when a educated person know, it’s not magic, it’s math.

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