

Complex Numbers: A Brief Introduction.

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Abstract

Complex numbers, although confusing at times, are one of the most elegant and interesting topics in mathematics to have surfaced in the last five centuries. Although it took time for them to catch on as legitimate mathematical tools, they have proven themselves again and again to be useful in a wide variety of math and engineering topics. This paper will provide a solid introduction to the world of complex numbers, including a thorough analysis of what exactly they are, and how one goes about operating with them. In addition to that, this paper will include some historical background, as well as an in-depth look into a few of the classic math problems that complex numbers have helped to solve, as well as some modern uses for complex calculations. The reader should be able to leave this paper with a better understanding and appreciation of the interesting and elegant world that is complex numbers.

Brief overview and basic terminology

To understand complex numbers, one must first grasp the idea of an imaginary number. In case your Algebra II knowledge is a little rusty, let me explain exactly what these things we call “imaginary numbers” are. The basis of imaginary number mathematics is the letter “ i ”. i is equal to the square-root of -1, ($\sqrt{-1}$). You may notice that this is an impossibility; square roots of negative numbers cannot exist. That is exactly where the idea of it being “imaginary” comes from. It is not a real number in that it cannot exist physically, but the magic comes in that it can be manipulated and used to find answers that have significance in the physical world. A complex number is a number that incorporates both real and imaginary elements, and is usually written in the form $a + bi$ where a and b are real numbers. These numbers are often times represented on a 2 dimensional grid; where the real element is represented on the x-axis, and the imaginary element is represented on the y-axis; therefore, a complex number can be represented by a point with coordinates (x,y) , as shown below in *Figure 1*. This method of geometrical representation is referred to as the “complex plane.”

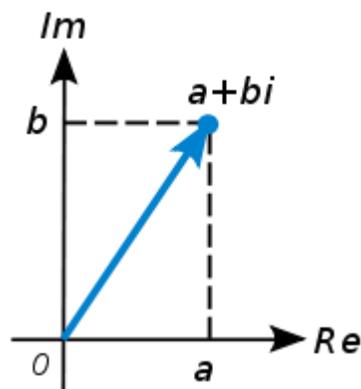


Figure 1. The Complex Plane with the point $(a + bi)$ plotted.

However, before we dive too deeply into the mathematics of complex numbers, let us first take a look at the history, and see how people first grappled with the idea of having an imaginary number. (Berlinghoff, 2002)

A brief history of imaginary numbers

Like many topics in math that are hard to conceptualize, imaginary numbers took a long time to become universally accepted as a legitimate concept of mathematical thought. The first time anyone really started to examine the possibility $\sqrt{-}$ actually being anything more than an impossibility was in the year 1545, when the great mathematician Cardano began working with solutions to quadratic and cubic equations. Although other mathematicians of his day had encountered scenarios where a certain formula led them to a square-root of a negative number, Cardano was the first one to not simply dismiss it as an absurd impossibility. A perfect example of this comes from his book *Ars Magna (1545)*. The specific problem is that of dividing 10 into two parts, whose product is 40. From the information given, we know that $x + y = 10$ and that $xy = 40$. With simple algebra, we can see that: $y = 10 - x$, and we can substitute that in to get the equation $x(10 - x) = 40$. By distributing x and adjusting the equation a bit, we see that $x^2 - 10x + 40 = 0$. By substituting the appropriate values into the quadratic equation, we get that

$$\frac{10 \pm \sqrt{100 - 160}}{2}$$

So our solutions are

$$x = \frac{10 + \sqrt{-60}}{2} \quad \text{and} \quad x = \frac{10 - \sqrt{-60}}{2}$$

With some manipulation we can simplify this to:

$$x = \frac{-5 \pm \sqrt{25 - 40}}{2} \quad \text{and} \quad x = \frac{-5 \pm \sqrt{25 - 40}}{2}$$

Which can be further simplified to:

$$x = \frac{-5 \pm \sqrt{5}}{2} \quad \text{and} \quad x = \frac{-5 \pm \sqrt{5}}{2}$$

And if we divide the entire numerator by 2, we get our final solutions of:

$$x = \frac{-5 \pm \sqrt{5}}{2} \quad \text{and} \quad x = \frac{-5 \pm \sqrt{5}}{2}$$

If you examine these solutions closely, you might realize that they do indeed satisfy the parameters of the problem; their sum is 10, because when you add the two solutions together, $(\frac{-5 + \sqrt{5}}{2} + \frac{-5 - \sqrt{5}}{2})$ just becomes equal to 0, and you are left with the simple equation $5 + 5$, which, as we can see, is equal to 10. The other parameter that we must meet is that their product needs to be equal to 40. To find this we must multiply the two solutions out as such:

$$\left(\frac{-5 + \sqrt{5}}{2}\right)\left(\frac{-5 - \sqrt{5}}{2}\right) = 25 - 5\sqrt{5} + 5\sqrt{5} - (\sqrt{5})^2$$

We can see that $(-5\sqrt{5} + 5\sqrt{5})$ is once again just equal to 0, so with that out of the equation, we are left with just: $25 - (\sqrt{5})^2$. Basic mathematics tells us that if we square a square-root, we are left with simply whatever is under the radical; in this case (-15). If we apply this to our remaining equation we get: $(25 - (-15))$ which we can see is indeed equal to 40. (Nahin, 1998)

You may be asking yourself how that is possible. From differential calculus, we can see that it is impossible for two numbers whose sum is ten to have a product of forty:

We know our two variables have to be x and $(10 - x)$, so if we set $f(x)$ equal to their product, we can use calculus to determine the maximum value that their product can be.

Since we know that $f(x) = x(10-x) = -x^2 + 10x$, we can determine that $f'(x) = -2x + 10$ and that $f''(x) = -2$.

If we set $f'(x)$ equal to 0, we can solve and see that $f'(x) = 0$ at the point $x = 5$. With the knowledge that $f'(5) = 0$ and that $f''(x) = -2$ at all values of x , we can say without doubt that $f(x)$ has its maximum at $x = 5$. If we substitute that value back into our original equation, we see that $f(5) = -(5^2) + 10(5) = (-25) + 50 = 25$. This proves that the maximum possible product of two numbers whose sum is 10 happens to be 25; 15 shy of our desired value of 40.

This proves that the solutions we found cannot exist as real numbers, but yet they fit the parameters of the problem. That is the magic of complex numbers; although they cannot themselves exist in the physical world, they enable us to find real solutions to real problems. Cardano was perplexed by this; he at first acknowledged that the solutions did fit the parameters of the problem, but he claimed that such solutions were useless when applied to real world problems. However, that assumption was soon proved wrong.

During the 16th century, one of the greatest problems plaguing mathematicians was finding a formula to calculate the roots of cubic equations. However, after years of different

mathematicians toiling away at it, a solution to the general cubic equation was finally discovered by none other than Cardano. Although other mathematicians had discovered solutions to particular types of cubic equations before, and those had aided Cardano significantly in his discovery of the general solution, Cardano is usually accredited with first publishing a solution to the general cubic, and even to this day the formula is still known as the “Cardano formula.” (Berlinghoff, 2002)

Cardano’s solution to cubic equations in the form of $x^3 + px + q = 0$ is:

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} - \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \quad (\text{Berlinghoff, 2002, p. 142})$$

The formula, in most cases, works just fine, and is able to provide you with a solution to any cubic equation in that form. However, certain equations present problems. One of these equations was proposed by the great mathematician Rafael Bombelli in his book *Algebra* (1572). The particular cubic equation he proposed was $x^3 - 15x - 4 = 0$.

From the equation, we see that our “p” value is (-15) and our “q” value is (-4). When we substitute those into Cardano’s formula, we get:

$$x = \sqrt[3]{-\frac{-4}{2} + \sqrt{\left(\frac{-4}{2}\right)^2 + \left(\frac{-15}{3}\right)^3}} - \sqrt[3]{-\frac{-4}{2} - \sqrt{\left(\frac{-4}{2}\right)^2 + \left(\frac{-15}{3}\right)^3}}$$

Which can be simplified to:

$$x = \sqrt[3]{2 + \sqrt{16 - 225}} + \sqrt[3]{2 - \sqrt{16 - 225}}$$

Which is equal to:

$$x = \sqrt{\frac{1}{2} + \frac{\sqrt{3}}{2}} + \sqrt{\frac{1}{2} - \frac{\sqrt{3}}{2}}$$

Again, we encounter a situation where we have to deal with the square-root of a negative number. You might think that maybe this equation simply does not have a real number root. However, if you go back and try $x = 4$ in the original equation, you will see that it indeed does satisfy the equation. How then, do we get an answer of 4 out of this jumbled mess of radicals? Cardano had encountered this problem himself, but had simply dismissed it as unsolvable. It was not until the work of Bombelli that people actually began doing advanced arithmetic with complex numbers. Bombelli was in fact the man who discovered how to do basic operations with complex numbers; but we'll get to that later. Anyway, if we incorporate i into the equation, we are able to work it out to achieve our desired answer of 4, as such:

$$x = \sqrt{\frac{1}{2} + \frac{\sqrt{3}}{2}} + \sqrt{\frac{1}{2} - \frac{\sqrt{3}}{2}}$$

Is equal to:

$$x = \sqrt{\frac{1}{2} + \frac{\sqrt{3}}{2}} + \sqrt{\frac{1}{2} - \frac{\sqrt{3}}{2}}$$

Which we can rewrite as:

$$x = \sqrt{\frac{1}{2} + \frac{\sqrt{3}}{2}} + \sqrt{\frac{1}{2} - \frac{\sqrt{3}}{2}}$$

Which we can say is equal to:

$$x = \sqrt[3]{2 + \sqrt{3}} + \sqrt[3]{2 - \sqrt{3}} \quad (\text{notice that the number under the radical is in the form of } (a + b))$$

Bombelli developed a method of calculating cube roots of these complex numbers, but seeing as his proof is long and somewhat tedious, we'll just state that $\sqrt[3]{2 + \sqrt{3}}$ and $\sqrt[3]{2 - \sqrt{3}}$ are equal to $(2 + \sqrt{3})$ and $(2 - \sqrt{3})$ respectively, which is indeed mathematically correct. Once we have it in this simple form, it is easy to see that with simple arithmetic, we are able to get our desired answer of 4, which satisfies our original equation of:

$$x^3 - 15x - 4 = 0.$$

$$x = \sqrt[3]{2 + \sqrt{3}} + \sqrt[3]{2 - \sqrt{3}} = (2 + \sqrt{3}) + (2 - \sqrt{3}) = 2 + \sqrt{3} + 2 - \sqrt{3} = 4$$

This was just the first of many instances of real situations where one needed to compute with complex numbers in order to obtain a real answer. (Nahin, 1998)

Basic operations

Although complex numbers must obey most of the same rules as real numbers, there are certain rules that we take for fact in the world of real numbers, but that don't hold as true in the world of complex numbers. The most common of these is the rule for multiplying radicals.

In the real number world, if we see something like $\sqrt{a} \sqrt{b}$, we can break it into $(\sqrt{a} * \sqrt{b})$, which is equal to \sqrt{ab} , which is much easier to work with. However, if we try to do the same thing with imaginary numbers, but in reverse, it doesn't work. For example:

$$3i * 2i = 6$$

In actuality:

$$3i * 2i = (3 \sqrt{-1}) * (2 \sqrt{-1})$$

Which can be simplified to:

$$(3i)(2i) = 6(i^2) = 6(-1) = -6$$

This just goes to show that although most basic operations with complex numbers are done with the same processes as are real numbers; there are certain situations and rules that don't carry over from one realm to the next. (Berlinghoff, 2002)

Simple operations can be done in essentially the same way that one would deal with any equation; by combining like terms and simplifying. That being said, one crucial component one must remember when operating with complex numbers is the powers of i . These properties of i are one of the things that make complex numbers so unique, as well as so useful in a variety of fields. The powers of i are cyclical in that as the power you raise i to increases, a repeating pattern appears in the values you get. Let's start from the bottom with i^0 . We know that anything raised to the 0 power (with the exception of 0 itself), is going to be equal to 1; i is no different: $i^0 = 1$. We also know that any number raised to the power of 1 is just going to be that number itself, unchanged. The same applies for $i^1 = i$.

i^2 is simply $(\sqrt{-1})^2$, and when you square a square-root, you are always left with what is under that radical, no matter what it is. By this definition, $i^2 = -1$.

i^3 can simply be thought of as $(i^2) * (i^1)$, which according to our definition of i^2 , is just $(-1) * (i)$, or just $-i$.

i^4 , in a similar manner, can be thought of as just $(i^2 * i^2)$, which again, according to our previous definition of i^2 , is just $(-1) * (-1)$, or just 1.

Now the pattern just begins to repeat itself: i^5 can be rewritten as just $(i^4) * (i^1)$ which we know now is just equal to $(1) * (i)$, or just i . The pattern continues from there, repeating itself with every fourth power. A general way to write this repeating pattern is simply:

$$i^{4n} = 1, \quad i^{4n+1} = i, \quad i^{4n+2} = -1 \quad \text{and} \quad i^{4n+3} = -i \quad \text{where 'n' is any integer. (Nahin, 1998)}$$

Once you know the powers of i , it is easy to do simple computations, such as multiplication. Take, for example, the general form of $(a + bi) (c + di)$:

If we multiply the two terms together, we get:

$$ac + ad + bc + bd(i^2)$$

We know that i^2 is just equal to (-1) , so if we substitute that into the equation, we get:

$$ac + ad + bc + bd(-1) = ac + ad + bc - bd$$

Then, if we factor out i from the appropriate terms, we are left with:

$$(ac - bd) + (ad + bc)i$$

This is indeed the product of our two original terms, and all complex numbers can be multiplied using this form. (Berlinghoff, 2002)

Some practical applications for complex numbers

As I've stated before, complex numbers have no real representation in the physical world, but yet they are an extremely useful tool in performing calculations that certainly have bearing in the real world. The most useful application for complex numbers is the fact that they can be geometrically represented on a 2 dimensional plane, and they have the capability of incorporating two different values into a single vector that can be used to perform computations. In other words; they essentially let us transform the traditional number line into 2 dimensions instead of just 1. This fact makes the complex number set essentially similar to the real number set, but instead of just going to infinity in one dimension, it goes to infinity in two dimensions. It has been proven that the cardinality of the complex set is equivalent to that of the real numbers, but the complex numbers are expressed as a two dimensional number system, which expands their useful applications dramatically.

In fields like electrical engineering, where in some cases a single number needs to represent multiple values, complex numbers prove themselves to be useful because you can use a single calculation to deal with a 2 dimensional problem. Adding complex vectors works exactly the same as computing with complex numbers, because the vectors created are just line segments going from the origin to a specified point. For example, if you need to add:

$(5+3i)$ and $(2-5i)$, then you need only to add the complex numbers as such:

$$(5 + 3i + 2 - 5i) = (7 - 2i) \quad (\text{Nahin, 1998})$$

This single step lets us add together vectors that incorporate two completely different values, thus letting us make computations such as that much more easily.

The complex plane is also used in the world of fractals and chaos theory. Probably the most famous fractal, the Mandelbrot set, is composed on the complex plane with a certain set of complex points, whose boundary makes up the perimeter of the fractal.

Conclusion

It cannot be denied that the advent of complex numbers has revolutionized the world of advanced mathematics, and it can undoubtedly be said that their usefulness in a variety of topics and applications has paved the way for numerous new innovations and discoveries.

As the French mathematician Jacques Hadamard (1865-1963) put it, "The shortest path between two truths in the real domain passes through the complex domain." (Berlinghoff, 2002, p. 146)

References

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